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Mitchell J. Nathan
Department of Educational Psychology/
Wisconsin Center for Education Research
University of Wisconsin–Madison
mnathan@wisc.edu

Ana C. Stephens
Department of Curriculum and Instruction/
Wisconsin Center for Education Research
University of Wisconsin–Madison
acstephens@wisc.edu

Martha W. Alibali
Department of Psychology/
Wisconsin Center for Education Research
University of Wisconsin–Madison
mwalibali@wisc.edu

Kenneth R. Koedinger
Carnegie Mellon University
koedinger@cmu.edu

Kate Masarik
San Diego State University
kmasarik@mail.sdsu.edu

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Enhancing Middle School Students' Representational Fluency: A Classroom-Based Study

**Mitchell J. Nathan, Martha W. Alibali, Kate Masarik,
Ana C. Stephens, and Kenneth R. Koedinger**

One important aspect of mathematical competence is the ability to reason with and among multiple representations. The importance of this skill, which we call *representational fluency*, is increasingly recognized by the mathematics education community as it struggles to reform algebra instruction and curricula. In the *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM; 2000) acknowledged that there has historically been a "preoccupation with number" (p. 211) and called for an increased focus on a variety of representations—including graphs, tables, and symbolic and verbal expressions—and the interconnections among them.

Members of the education research community have trumpeted both the need for and the benefits of making these connections (e.g., Ainsworth, 1999; Brenner et al., 1997; Confrey & Maloney, 1996; Knuth, 2000). In demonstrating the need for representational fluency, Knuth (2000) showed that advanced high school algebra students who are familiar with equations and graphs do not readily connect graphical representations such as the Cartesian coordinate system to their knowledge of equations. Students also fail to use graphs even when graphical solutions are easier and more efficient. Knuth suggested this may be due to an almost exclusive curricular focus on symbolic representations and manipulations. Moschkovich, Schoenfeld, and Arcavi (1993) argued that competence in the mathematics of functions depends on moving flexibly among representations. They went so far as to use the connections made among representations as the criterion for evaluating curricula and student assessments. At the same time, they documented just how rare this level of competence is among students—and how difficult to achieve.

The benefits of representational fluency are considered to be far-reaching. Kaput (1989) suggested that mathematical meaning making is actually built upon the ability to translate within and among representations and that meaning is fundamentally based on a "relational semantics" between "linking representations" (p. 168), including internal mental representations and physical systems as well as tables, symbols, and graphs. Similarly, Lesh and colleagues (Lesh & Doerr, 2003a, 2003b; Lesh & Lehrer, 2003) have placed representational fluency on a par with symbol manipulation as a core competency in mathematical thinking. Its importance extends beyond the classroom into the workplace, where it is seen as an essential skill, mediating data-driven decision making, mathematical and scientific modeling, the interpretation and explanation of complex systems, and the use of increasingly advanced technology (Dark, 2003; English, 2007; Lesh, Zawojewski, & Carmona, 2003).

Research from a psychological perspective has also shown important benefits of multiple representations for learning and performance. Evidence from both behavioral research (e.g., Griffin, Case, & Siegler, 1994; Stenning & Oberlander, 1995) and neuroscience (e.g., Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999) has pointed to a dual system of linguistic and spatial representations that supports mathematical reasoning. Tabachneck and colleagues (Tabachneck, 1992; Tabachneck, Leonardo, & Simon, 1994; Tabachneck-Schijf, Leonardo, & Simon, 1997) showed how an expert in economics achieved an understanding of an economic situation out of the reach of novices by combining graphical and verbal representations. Schwartz (1995) found that the availability of multiple representations played a key role in students' generation of abstract representations. Students working in dyads had a need to communicate across the multiple representations produced by dyad members and were therefore more likely to generate abstract representations than matched individuals working alone.

Despite the benefits of and need for representational fluency, little is known about the performance trade-offs of different representations used by mathematics students or about students' ability to move among various representational formats during problem solving and mathematical reasoning. Furthermore, it is not clear how best to foster representational fluency through instruction. Prior investigations in this area have been limited in their applicability to K–12 educational settings, usually emphasizing study of more mature students or adults, typically in laboratory and pullout settings (see Grouws, 1992, for a review). One important exception was an in-class, one-month intervention conducted by Brenner and her colleagues (1997) with English- and Spanish-speaking junior high school students. The intervention emphasized translation among different representations (tables, graphs, words, equations) as students discussed, represented, and solved problems with meaningful contexts in a collaborative, guided discovery setting. Students using the experimental curriculum improved more than comparison students on a variety of word problem-solving tasks and measures of representation use, regardless of language status.

There is also a dearth of information about students' preinstructional knowledge and reasoning about representation use. For example, Swafford and Langrall (2000) pointed out, "Although there has been extensive research on algebra learning, educators do not have a complete picture of what students can do in algebra prior to formal instruction" (p. 89). Recently, however, Nathan and Kim (2007) reported findings from cross-sectional (N = 372) and longitudinal (N = 81) analyses of U.S. middle school students' performance on pattern generalization tasks of linear functions using graphical and verbal representations. In addition to the expected findings—grade-level differences and superior performance on near predictions over far predictions and on far predictions over abstraction tasks—they found an advantage for combining representations. Specifically, student performance was higher when patterns were presented in both graphical form (points or lines) and words (verbal rules or lists of instances) than when they were presented in only one form or the other. The advantage was even evident among sixth graders who had not yet received formal instruction in algebra. This finding of synergy across representations provides further impetus for exploring the nature and effects of representational fluency.

Objectives of the Study

The objective of the classroom study reported here was to document middle school students' preinstructional understanding of the representations commonly used in beginning algebra education and to show how these understandings change after instruction. Our study included a novel curricular approach—which we call *Bridging Instruction*—that specifically draws on students' mathematical preconceptions and invented solution strategies as a basis for enhancing their representational fluency across a range of measures. Our investigation was prompted by the need for education research that is both valid and generalizable (e.g., National

Research Council [NRC], 2002; Schoenfeld, 1999). To address ecological validity, we situated our investigation within the regular day-to-day constraints of the classroom over a 9-week period, with instruction carried out by the regular classroom teacher, and compared the performance of students in the experimental classes with that of students using *Connected Mathematics* (Lappan et al., 1998a, 1998b), a highly regarded, reform-based curriculum that was regularly used by the classroom teacher. *Connected Mathematics* draws on contemporary learning theory (e.g., NRC, 1999; Resnick, 1987). It takes a constructivist approach that emphasizes collaborative, discussion-oriented activities that use data gathering and representation as well as problem solving and investigation to make mathematical representations and procedures conceptually meaningful to students. *Connected Mathematics* is used throughout the U.S. and has been shown to be an effective curriculum for teaching middle school mathematics (Hoover, Zawojewski, & Ridgway, 1997).

We acknowledge that many factors distinguish the two instructional conditions under investigation and do not claim that our research design isolates one single factor across conditions. Rather, our approach is informed by design-based research methodology (also referred to as *design experiments* and *teaching experiments*; Brown, 1992; Cobb, 2000; Collins, 1992; Design-Based Research Collective, 2003), which offers a principled way of addressing the relationship between a theoretically guided intervention and its outcomes. Design-based research provides for flexibility of interventions—what Koedinger (2002) referred to as “the hare of intuitive design” (p. 8)—and can be complementary to the incremental approach of experimental design—Koedinger’s “tortoise of cumulative science” (p. 8). By using identical measures of performance across the two instructional conditions and having the same teacher enact the two curricula, we sought to provide a strong basis for comparison even in the face of the wide-ranging differences that are inevitable when implementing two innovative and complex classroom interventions.

The Bridging Instruction *Curriculum*

Our intent was to explore the feasibility of *Bridging Instruction* as an approach to algebra education that draws explicitly on students’ mathematical knowledge, preinstructional intuitions about quantities and relations, and invented solution strategies as the basis for developing facility with formal representations and procedures. Approaches of this sort follow the instructional implications of social constructivism (e.g., Resnick, 1987)—specifically:

1. That reasoning and learning are knowledge-driven and socially based;

2. That learners draw heavily on the use and mastery of artifacts, such as tools, and the canonical representations of a discipline or profession; and

3. That much initial learning is highly situation- and discipline-specific and tends to require a great deal of mental effort, time, and external support before it becomes more generalized.

Instructional approaches from within this perspective target the active involvement of learners in collaborative investigations within discourse-rich social settings (e.g., Brown & Campione, 1994; Cognition and Technology Group at Vanderbilt, 1997), as well as the relation of learners’ preexisting knowledge to the concepts of curricular interest. By grounding (e.g., Lakoff & Nunez, 2000) new mathematical ideas and representations in students' own forms of quantitative reasoning, we hoped to imbue abstract, algebraic formalisms with meaning and thereby enhance students' abilities to reason with and among various algebraic representations.

One of the central claims of social constructivist approaches to mathematics education is that students' prior conceptions of mathematics can be used to shape curricular progression. Lehrer, Strom, and Confrey (2002), for example, showed that use of elementary students' initial understandings of object relations and object similarity—drawn from iconic, indexical, and symbolic perspectives—supported the development of mathematical generalizations across algebraic and geometric forms of reasoning and representation. Nathan and Koedinger (2000) demonstrated that middle school students' invented solution methods for algebra story problems were effective entry points for teaching a generalized modeling approach to algebra. In that study, students exhibited and talked about notions of inverting operations, describing relations in terms of unknown quantities and using covariation to show changes in output values as a function of changes in input values—all using invented terminology and strategies. Careful pedagogical connections were then made and publicly displayed in the classroom to promote formal approaches to algebraic reasoning. We drew on these earlier studies in designing a 9-week intervention that allowed for a broad and rigorous assessment of the *Bridging Instruction* approach.

The Representation-Complexity Trade-Off in Problem Solving

Problem-solving performance is a key aspect of representational fluency, offering an indication of students' abilities to reason analytically within a single representational format. Research has shown that students' problem-solving performance is influenced by the representational format in which problems are presented (Brenner et al., 1997; Knuth, 2000; Nathan & Kim, 2007). For example, in a study by Koedinger, Alibali, and Nathan (2008), low- and high-performing college students (n_1 = 153, n_2 = 65, respectively) solved relatively simple (2-step) linear problems with greater accuracy when the problems were presented in words (as stories or verbal equations). However, for students in both samples, symbolic representations were superior for solving the most complex problems with multiple occurrences of an unknown. Apparently, as the complexity of the quantitative relations increased, the verbal representations did not "scale" as well as symbolic representations; rather, the verbal representations became difficult to comprehend and manipulate. This representation-complexity trade-off has also been found using positive and negative numbers with primary grade students (Verzoni & Koedinger, 1997).

Koedinger and colleagues (2008) suggested that verbal representations in algebra might serve as the conceptual ground for learning generalizable, formal symbolic structures that apply more readily to more complex quantitative relations. In the current investigation, we expressly manipulated the representational format used to present problem-solving tasks. We looked to replicate the representation-complexity trade-off documented by Koedinger and colleagues, but also to extend it to the range of mathematical representations that are now commonplace in algebra curricula.

Translation Across Mathematical Representations

Translation across mathematical representations constitutes another key aspect of representational fluency (Kaput, 1989; Moschkovich et al., 1993). Whereas problem solving depends largely on students' abilities to comprehend and manipulate a *given* representation (Koedinger & Nathan, 2004), translation requires students to generate a mathematical representation of a *new* type, given information depicted in a different form. Previous research on mathematical reasoning has largely been limited to the study of problem-solving performance and typically has not directly measured translation. Because of its importance in understanding representational fluency and the relative scarcity of studies in this area, we made translation a central consideration in the current investigation. Furthermore, in keeping with advances in current algebra reform, we assessed students' translation abilities as they moved to and from a broad range of representations, including words, tables of values, and graphs as well as symbolic equations.

Research Questions

Our first research question focuses on the impact of the two curricula on students' overall performance. The following two questions target the ways in which the specific performance improvements manifest in the two measures of representational fluency: representational problem solving and translation.

1. *How do beginning algebra students' performance improvements with* Bridging Instruction *compare to their performance improvements with* Connected Mathematics*?* Prior research suggested that student learning of algebraic representations and reasoning would be substantial using *Connected Mathematics*. In selecting this formidable control condition, we wanted to explore the additional benefits *Building Instruction* might provide by tying instruction more directly to students' informal mathematical ideas.

2. *What are the initial successes and challenges that beginning algebra students demonstrate when asked to translate from one mathematical representation to another?* We addressed this question by using assessment items that measured students' ability to generate a cued output representation (symbolic equation, graph, table, or words) when given a mathematical pattern in an input representation (symbolic equation, graph, table, or words).

3. *Do beginning algebra students exhibit the representation-complexity trade-off during problem solving?* As previously noted, researchers have found that verbal representations support higher performance for relatively low-complexity tasks, whereas symbolic representations are superior for high-complexity tasks. In this study, we sought to replicate this effect and extend it to tables and graphs. The low- and high-complexity tasks involved linear and nonlinear functions, respectively.

These questions guided the ensuing investigation and were used to organize the empirical results.

Method

Participants

Ninety seventh- and eighth-grade students in four mathematics classrooms in a middle class school district in the Midwestern U.S. participated in this study. Eight students were excluded from the final analyses because they switched classrooms during the study in a way that altered their condition assignment. Of the remaining 82 students, 70 took both the pretest and posttest assessments and therefore were included in all analyses that involved participants as the unit of analysis. An additional 12 students took only one of the two assessments; data from these 12 students were included in analyses that involved items as the unit of analysis. Thus, the item analyses are based on the performance of 82 students.

Students were placed in one of four combined Grade 7/8 classes. Consistent with district policy and practice, students were assigned to mathematics classes based on consultation with their parents and the mathematics teacher from the prior year and thus were not randomly assigned to condition. The regular Grade 7/8 classroom teacher taught all four classes. Two of the classrooms, designated as controls, implemented a *Connected Mathematics* (CM) curriculum designed around the seventh- and eighth-grade series (Lappan et al., 1998a, 1998b). The other two classroom sections were designated experimental classes and implemented the *Bridging Instruction* (BI) approach. The teacher determined which curriculum was assigned to each class section.

Materials

The materials used in this study consisted of the curriculum, which varied for the CM and BI conditions, and the translation and problem-solving assessment instruments used in both conditions. The assessments were administered twice to all students, just before and just after the 9-week intervention.

Curriculum differences. The two instructional conditions had some commonalities as well as some important differences that led us to hypothesize concomitant differences in student outcome measures. The CM curriculum had been in regular use by the mathematics teacher in previous years and so served as the baseline condition for algebra learning. The curriculum is well described elsewhere (Lappan et al., 1998a, 1998b) and commercially available. During the 9-week algebra unit, CM students worked primarily on activities in the seventh-grade units *Variables and Patterns* and *Moving Straight Ahead* and the eighth-grade units *Thinking with Mathematical Models*; *Frogs, Fleas, and Painted Cubes*; and *Say It with Symbols*. CM addresses how words, tables, graphs, and algebraic symbols can depict data for linear and nonlinear relations and how these representations are interrelated. The CM algebra strand (CM Project, 2006a) targets, among other things, the following four competencies:

- Using tables, graphs, symbolic expressions, and verbal descriptions to describe and predict patterns of change in variables;

- Recognizing, in various representational forms, patterns of change associated with linear, exponential, and quadratic functions;

- Using numeric, graphic, and symbolic strategies to solve problems involving linear, exponential, and quadratic functions; and
- Solving linear and simple quadratic equations by manipulating symbols and using tables and graphs.

Across its strands, CM emphasizes “moving flexibly among graphic, numeric, symbolic, and verbal representations and recognizing the importance of having various representations of information in a situation” (CM Project, 2006b). Because of this emphasis, CM seemed an appropriate comparison curriculum for our study of representational fluency. Table 1 shows the sequence of topics covered in the CM curriculum as it was administered during the 9-week study.

Table 1
Outline of CM Curriculum (Control Condition) During the 9-Week Intervention

Unit	Content
Bridges and Pennies (from *Moving Straight Ahead*)	• Data collection and organization (tables) • Graphing and pattern generalization • Equation writing • Linear pattern generalization
Introducing Graphs, Tables, and Verbal Rules (from *Variables and Patterns*)	• Graphing of linear functions and relations • Analysis of linear graphs and tables • Translation of graphs and tables to verbal rules • Translation of verbal rules to equations
Identifying Linear Patterns and Introduction to Solving Linear Equations (from *Moving Straight Ahead*)	• Linear relationships • Recognition and representation of linear relationships in tables, graphs, words, and symbols • Solution of simple linear equations
Solving Linear Equations and Systems of Equations (from *Moving Straight Ahead*)	• Use of the graphing calculator to solve linear equations and systems • Solution of linear equations using the symbolic and “undoing” methods
Introduction to Linear Functions (from *Thinking with Mathematical Models* and *Say It with Symbols*)	• Representation of relationships • Introduction to functions and modeling • Slope • Determination of the equation of a line

Unit	**Content**
Nonlinear Patterns and Functions (from *Growing, Growing, Growing: Exponential Growth and Decay*)	• Splitting and doubling • Graph construction • Equation writing and comparisons • Pattern generalization
Painted Cube Problem (from *Frogs, Fleas, and Painted Cubes*)	• Tables of values • Verbal rules • Comparison of constant, linear, and nonlinear (quadratic and cubic) functions • Use of the graphing calculator (TI-82) to represent linear and nonlinear patterns and solve linear and nonlinear functions • Equation writing for linear and nonlinear functions • Reading of graphs of linear and nonlinear functions • Pattern generalization and prediction with linear and nonlinear functions

Like CM, BI emphasizes representational fluency. However, as implemented in this study BI differed from CM in at least two respects. First, the teacher was explicitly directed to elicit students' intuitive mathematical notions and invented solution strategies and representations as starting points for instruction and to overtly connect these preconceptions to formal solution methods and representations. Second, linear and nonlinear functions were compared and contrasted much earlier than in the CM condition—starting with the first topic, *Bridges and Pennies*—to avoid establishing as default properties the peculiarities of linear functions that do not generalize to other functions (Schwarz & Hershkowitz, 1999). The teacher frequently contrasted the behavior and the representations of linear and nonlinear relations to make the properties of each more salient to learners. As a consequence, the teacher ended up providing more class sessions directly addressing nonlinear relations to BI than to CM students over the 9-week period (16 out of 38 sessions for BI, as compared with 10 out of 38 for CM). Since the number of sessions was the same in the two conditions, BI students spent less time exclusively on linear relations. Table 2 shows the sequence of topics covered in the BI curriculum.

Table 2
Outline of BI Curriculum (Experimental Condition) During the 9-Week Intervention

Unit	**Content**
Bridges and Pennies (from *Moving Straight Ahead*)	• Data collection and organization (tables) • Graphing and pattern generalization of linear and nonlinear functions • Translation of verbal rules to symbolic equations for linear functions • Criteria for a good representation • Linear and nonlinear pattern generalization
Law of Torque	• Linear and nonlinear (quadratic) relations • Translation of verbal rules of nonlinear patterns to graphs
Building Squares	• Perimeter and area as linear and nonlinear functions • Grounding the meaning of graphs • Representation of linear and nonlinear patterns as verbal rules and equations • Writing and solution of quadratic equations for areas of rectangles

Unit	**Content**
Word Problem Solving (some activities from CM *Moving Straight Ahead*)	• Construction and interpretation of tables and graphs of linear and nonlinear relations • Relation of verbal strategies to symbolic modeling and problem solving • Foot race problem with handicapping (from CM *Moving Straight Ahead*) • Graphs and intersecting lines • Systems of linear equations • Story problem solving with linear and nonlinear functions
Nonlinear Patterns and Functions (from *Growing, Growing, Growing: Exponential Growth and Decay*)	• Splitting and doubling • Graph construction • Relation of linear and nonlinear patterns • Equation writing and comparisons of $2x$, x^2, 2^x, and $2-x$ • Translation of symbol to table, symbol to word for linear and nonlinear functions • Pattern generalization and prediction with linear and nonlinear functions
Painted Cube Problem (from *Frogs, Fleas, and Painted Cubes*)	• Translation of tables of values to verbal rules for linear and nonlinear functions • Relation of constant, linear, and nonlinear (quadratic and cubic) functions • Translation of verbal rules to graphs and equations for linear and nonlinear functions • Use of the graphing calculator (TI-82) to represent linear and nonlinear patterns and solve linear and nonlinear functions • Equation writing and substitution for linear and nonlinear functions • Pattern generalization and prediction with linear and nonlinear functions

Throughout the 9-week intervention, BI students were first called upon to decide for themselves, individually or in small groups, how they would represent relationships or solve for unknown values. Thus, students' intuitive notions of how to organize data derived from linear and nonlinear relationships (including both quadratic and exponential functions)—and depict them pictorially, verbally, and through arithmetic procedures and relations—served as precursors to targeted activities and instruction on the standard use of tables, graphs, and equations.

Representing and reasoning about exponential relations were presented to BI students in the context of splitting and doubling, based on the work of Confrey and Smith (1995). The graphing of quadratic and cubic functions was taught using Kalchman's method of recomposing manipulatives (Kalchman, 1998; Kalchman, Moss, & Case, 2001). Students used tiles to match an area and then decomposed the squares by stacking the area tiles along grids of 1-inch graph paper and marking off the values (heights) that showed the number of square inches covered by each square. This provided a concrete way for students to record the area (or perimeter) of each square and to see how these measures could be translated from numerical representations to the height on a graph. This idea was further expanded to the idea of describing the volume of blocks in the cube problem. BI students came up with systems, such as color-coding schemes, to provide a proper accounting of the linear, area, and volumetric measures that would support predictions about the characteristics of novel cubes.

Assessment design. The items used on the pretest and posttest assessments were drawn from a universe of 72 items that were generated using a $2 \times 2 \times 2 \times 9$ factorial design. The factors were item linearity (linear or nonlinear), slope sign (increasing or decreasing), and representation input-output pair (graph to symbol, graph to table, graph to word, symbol to graph, symbol to table, symbol to word, word to graph, word to symbol, word to table). For each combination of linearity and slope sign, there were two cover stories. For example, all linear increasing problems featured either Luke riding a scooter and keeping track of the distance he traveled or Cassandra selling phone cards and keeping track of the relationship between the cost of the card and the number of minutes available on the card. This design yielded a total of 72 different items. These 72 items were distributed among 12 different forms, with three linear and three nonlinear and three increasing and three decreasing items per form. In each item, a scenario was presented in words, along with a function presented in an input representation (i.e., a graph, symbolic equation, or word expression).

Students were asked to respond to three parts. The first two were designed to assess students' abilities to solve problems within the given representation. On linear problems, the first part asked students to use the input representation to find a specific value of the dependent variable given a specific value of the independent variable, while the second part asked students to find a specific value of the independent variable given a specific value of the dependent variable. On nonlinear problems, the first part asked students to use the input representation to find a specific value of the dependent variable given a specific non-zero value of the independent variable, while the second part asked students to find the value of the dependent variable when the value of the independent variable was zero. (See the appendix for examples of both linear and nonlinear functions.)

The third part was designed to assess students' abilities to translate from one representation to another. In this part, students were asked to represent the function used in the item in a representation different from the one initially presented.

Procedure

Students in all four classes were given the written assessments at pretest and then again at posttest, approximately 10 weeks later. Each assessment form consisted of six problems that tested students' problem-solving abilities (solving for an unknown value) and their abilities to translate from one mathematical representation to another. The regular classroom teacher administered the assessment instrument during the normal time scheduled for students' mathematics class. Students were given 35 minutes to complete the examinations and were allowed to use calculators.

Scoring and Coding

Problem solving. As described above, in the problem-solving parts of the items students were asked to find a specific value of the dependent variable given a specific value of the independent variable, and vice versa. Each solution was scored as correct or incorrect. For problems that involved symbolic equations or word expressions, answers were coded as correct only if they were exact, with the exception of one problem for which the story context permitted rounding. For problems that involved graphs, answers were coded as correct if they fell within ⅛ inch of the correct value on the coordinate system. Such error was considered acceptable because students were not given corresponding algebraic equations along with the graphs and thus had to rely solely on the graphs themselves to find solutions.

Translation. As noted, the translation parts of the items asked students to create an output (a graph, symbolic equation, table, or word expression) containing information mathematically equivalent to that given in the input representation presented in the item. Each translation was scored as correct or incorrect. Given the diverse nature of the outputs, separate criteria were established for each:

- *Graphs.* To qualify as correct, graphs needed to have at least three correct and no incorrect data points. It was not necessary for the axes to be labeled with words or for the points to be connected with a line or curve.

- *Symbols.* To qualify as correct, symbolic equations needed to be completely accurate. It was acceptable for words to be used in place of variables (e.g., distance = 8 × min).

- *Tables.* To qualify as correct, table outputs needed to have at least three accurate entries (ordered pairs) and no inaccurate entries. Entries needed to be exact with the exception of those generated from graphs, for which the previously described ⅛-inch rule was employed. It was not necessary for table columns to be labeled.

- *Word expressions.* To qualify as correct, word expressions needed to accurately describe the computational strategy necessary to solve the given problem, either entirely in words or in a word-symbol combination.

Results

We analyzed problem-solving and translation performance separately using repeated measures analysis of variance (ANOVA), conducting analyses at both the participant level and the item level. Unless otherwise noted, all reported results are significant with an alpha level of 0.05. The slope-sign factor was not involved in any significant main effects or interactions, and it was therefore collapsed in the analyses reported here.

Problem Solving

For the problem-solving data, the participant analysis utilized a 2 × 2 × 2 (curriculum × test date × linearity) repeated measures ANOVA, with curriculum (experimental or control) as a between-participants factor, and test date (pre or post) and linearity (linear or nonlinear) as within-participant factors. Because representation was not balanced across participants, it was not included as a factor in the participant analysis. The item analysis utilized a 2 × 2 × 2 × 3 (curriculum × test date × linearity × representation) ANOVA, with linearity (linear or nonlinear) as a between-items factor, and curriculum (experimental or control), test date (pre or post), and representation (word, graph, or symbol) as within-items factors. Simple effects were analyzed based on the procedures described by Keppel (1991).

The participant analysis revealed significant main effects of linearity, $F(1, 68) = 69.54$, $\eta^2 = .50$, and test date, $F(1, 68) = 11.62$, $\eta^2 = .14$, as well as a significant interaction of test date × curriculum, $F(1, 68) = 4.01$, $\eta^2 = .048$. These effects were also significant in the item analysis: main effect of linearity, $F(1, 14) = 16.10$, $\eta^2 = .53$; main effect of test date, $F(1, 14) = 5.58$, $\eta^2 = .28$; and interaction of test date × curriculum, $F(1, 14) = 9.33$, $\eta^2 = .40$. The item analysis also revealed a significant main effect of representation, $F(2, 28) = 117.66$, $\eta^2 = .77$, as well as significant interactions of representation × linearity, $F(2, 28) = 20.46$, $\eta^2 = .13$; representation × test date, $F(2, 28) = 6.36$, $\eta^2 = .26$; and representation × linearity × test date, $F(2, 28) = 4.61$, $\eta^2 = .19$. We consider each of these effects in turn.

As expected, students performed better overall at posttest than at pretest, and better on linear (low-complexity) problems than on nonlinear (high-complexity) problems. They also performed better on graph items than on symbol or word items. However, these main effects were qualified by significant interactions among the factors. Figure 1 presents the representation × linearity × test date interaction. Linearity did not influence student performance when problems were presented in graphs, but it did influence performance when problems were presented in words: pretest $F(1, 14) = 43.95$, posttest $F(1, 14) = 28.52$. For problems presented in symbols, linearity did not influence performance at pretest, but it did influence performance at posttest, $F(1, 14) = 6.25$, because of isolated improvements in performance on linear symbolic problems.

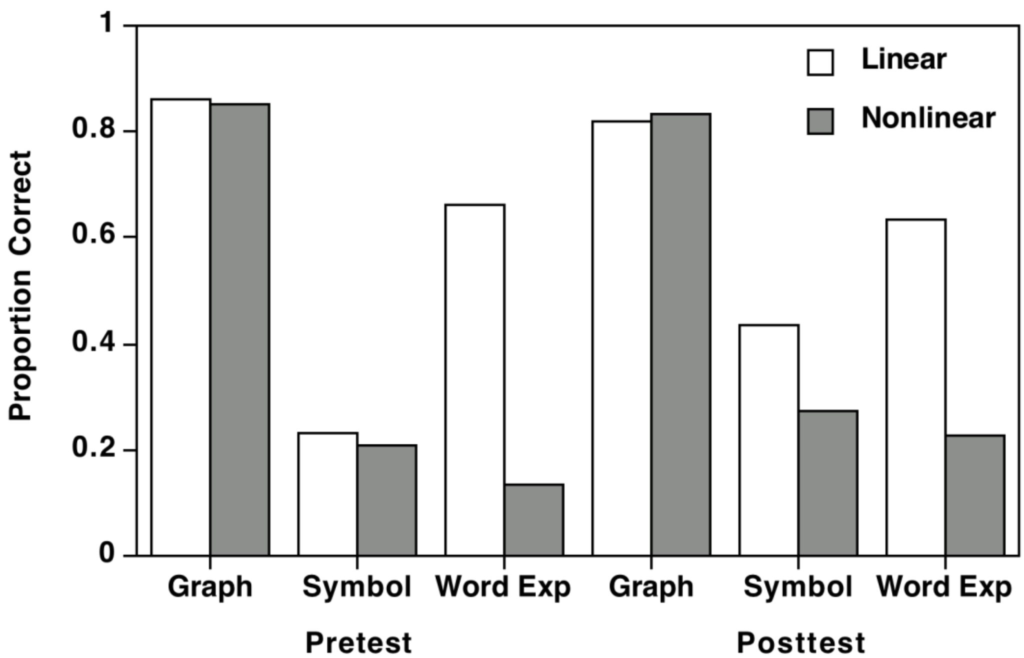

Figure 1. **Proportion of problem-solving items solved correctly by test date, representation, and linearity. Effects are averaged over both instructional approaches.**

Examining this interaction from a different perspective, students succeeded more often on linear problems that were presented in words than on those that were presented in symbols, whereas on nonlinear problems the reverse was found to be true, though to a lesser degree, yielding a significant interaction of the word-symbol contrast with linearity, $F(1, 14) = 29.81$. These results lend support to previous findings that document a verbal advantage for problems of relatively low complexity and a symbolic advantage for problems of relatively high complexity (Koedinger et al., 2008). Koedinger and colleagues' study showed that, given a fairly simple linear problem situation, students succeeded more often with problems presented verbally (i.e., as word equations or story problems) than with those presented in symbols. However, for more complex problems, students performed better when problems were presented symbolically (i.e., as algebraic equations). The present results replicate these earlier findings. The results are also extended to include graphical representations. As with verbal representations, graphs are highly effective for low-complexity tasks, and like equations, graphs scale nicely as complexity increases. We elaborate on this dual nature of graphs in the Discussion section.

The comparison of the two curricula is captured in the curriculum × test date interaction, which was significant in both the participant and item analyses, as noted above. Students in the experimental (BI) group made greater gains from pretest to posttest than did students in the control (CM) group, $F(1, 15) = 9.86$. Performance in the control group held steady, with 52.2% correct (SD = 34.3%) at pretest and 50.6% correct (SD = 28.4%) at posttest, whereas performance in the experimental group improved from 45.9% correct (SD = 36.1%) at pretest to 56.8% correct (SD = 31.3%) at posttest.

As seen in Figure 2, pretest to posttest gains for students in the experimental classes were relatively broad, with improvement dispersed across both levels of linearity (collapsed across representations), linear $F(1, 7) = 6.45$, nonlinear $F(1, 7) = 5.15$, and across both verbal and symbolic problem representations (collapsed across linearity), verbal $F(1, 15) = 8.88$, symbolic $F(1, 15) = 3.96$, $p = .07$. This pattern is consistent with the attention the experimental instruction gave to exposing students to multiple mathematical representations and patterns of change. These gains also make clear that seventh- and eighth-grade students can in fact improve their algebraic reasoning about nonlinear functions in various representational formats.

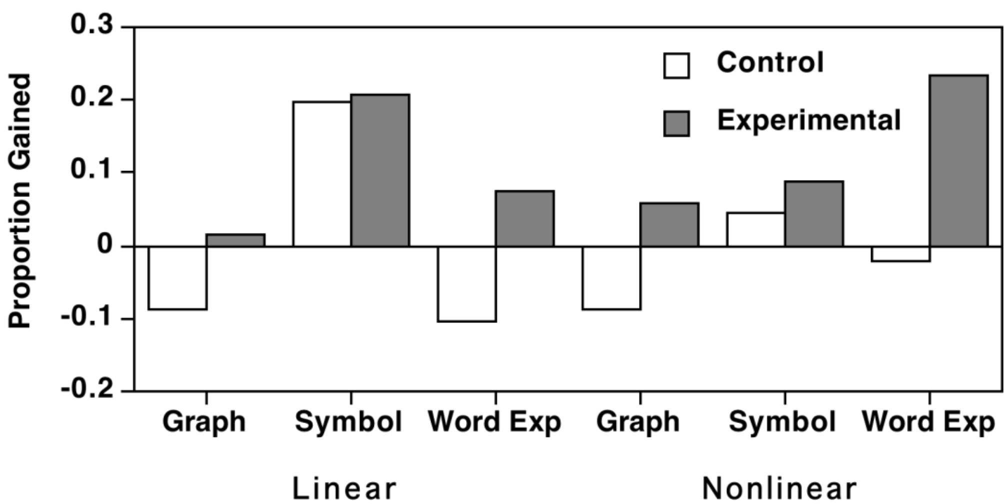

Figure 2. **Control (CM) and experimental (BI) group problem-solving gains on graph, symbol, and word expression representations by linearity.**

In contrast, students in the control classes did not make substantial gains on either linear or nonlinear problems (collapsed across representations). They made significant gains on symbolic problem representations (collapsed across linearity), $F(1, 15) = 8.18$, but also showed significant losses on graph problem representations, $F(1, 15) = 4.80$.

As both curricula were implemented over the same 9-week period, it is clear that the experimental curriculum devoted less time to linear functions than the control curriculum. However, despite the control curriculum's greater emphasis on linear situations (Table 1) and the experimental curriculum's more dispersed focus, gains on linear problems were actually greater for the experimental group. Overall, the control group held steady on linear problems from pretest to posttest (pretest $M = 62.0\%$, $SD = 31.2\%$; posttest $M = 61.3\%$, $SD = 22.2\%$), with the gains on linear symbolic problems offset by losses on linear graph problems. In contrast, the experimental group made marginally significant gains on linear problems overall (pretest $M = 54.7\%$, $SD = 33.2\%$; posttest $M = 64.8\%$, $SD = 26.4\%$), $F(1, 7) = 4.08$, $p = .08$. The experimental group also made significant gains in solving nonlinear problems (pretest $M = 37.0\%$, $SD = 37.3\%$; posttest $M = 48.8\%$, $SD = 34.2\%$), $F(1, 7) = 7.67$.

Translation

For the translation tasks, the participant analysis utilized a 2 × 2 × 2 (curriculum × test date × linearity) repeated measures ANOVA, with curriculum (experimental or control) as a between-participants factor, and test date (pre or post) and linearity (linear or nonlinear) as within-participant factors. Because input-output pair was not balanced across participants, it was not included as a factor in the participant analysis.

The item analysis utilized a 2 × 2 × 2 × 9 (curriculum × test date × linearity × input-output pair) ANOVA, with linearity (linear or nonlinear) as a between-items factor, and curriculum (BI or CM), test date (pre or post), and input-output pair (graph to symbol, graph to table, graph to word, symbol to graph, symbol to table, symbol to word, word to graph, word to symbol, or word to table) as within-items factors.

If the BI curriculum fostered representational fluency to a greater degree than CM, this would produce a test date × curriculum interaction. In addition, we would anticipate an advantage for linear (low-complexity) functions over nonlinear functions, and a general improvement from pretest to posttest.

The participant analysis revealed significant main effects of linearity, $F(1, 68) = 23.27$, $\eta^2 = .25$, and test date, $F(1, 68) = 42.86$, $\eta^2 = .37$, as well as a significant interaction of test date × curriculum, $F(1, 68) = 5.94$, $\eta^2 = .051$. These effects were also significant in the item analysis: main effect of linearity, $F(1, 6) = 19.73$, $\eta^2 = .77$; main effect of test date, $F(1, 6) = 16.85$, $\eta^2 = .73$; and interaction of test date × curriculum, $F(1, 6) = 8.42$, $\eta^2 = .49$. The item analysis also uncovered significant main effects of curriculum, $F(1, 6) = 6.29$, $\eta^2 = .36$, and input-output pair, $F(8, 48) = 8.27$, $\eta^2 = .48$, as well as significant interactions of input-output pair × linearity, $F(8, 48) = 2.80$, $\eta^2 = .16$, and input-output pair × test date, $F(8, 48) = 2.58$, $\eta^2 = .27$. We consider each of these effects in turn.

As expected, students performed better overall on translation problems at posttest than at pretest. The comparison of the two curricula is captured in the curriculum × test date interaction, which was significant in both the participant and item analyses, as noted above. Students in the BI group made greater improvements from pretest to posttest than did students in the CM group, $F(1, 7) = 10.24$. The control group demonstrated a 5.6% increase in correct responses, whereas the experimental group demonstrated a 17.6% increase.

Overall, students performed better on linear (low-complexity) problems than on nonlinear (high-complexity) problems. Performance also varied across the input-output pairs. However, these main effects were qualified by significant two-way interactions of input-output pair with linearity and test date. Table 3 presents pretest and posttest data, as well as gain scores, for each input-output pair and each level of linearity.

Table 3
Proportion of Participants (Collapsed Across Conditions) Who Succeeded on Each Input-Output Pair at Pretest and Posttest

	Pretest Mean (*N*)	**Posttest Mean (*N*)**
Linear problems		
Graph to symbol	0.00 (22)	0.00 (27)
Graph to table	0.27 (26)	0.40 (25)
Graph to word	0.00 (27)	0.04 (25)
Symbol to graph	0.00 (22)	0.22 (27)
Symbol to table	0.04 (27)	0.44 (25)
Symbol to word	0.15 (26)	0.28 (25)
Word to graph	0.13 (22)	0.30 (27)
Word to symbol	0.39 (26)	0.40 (25)
Word to table	0.33 (27)	0.48 (25)
Nonlinear problems		
Graph to symbol	0.00 (27)	0.00 (25)
Graph to table	0.09 (22)	0.44 (27)
Graph to word	0.00 (26)	0.00 (25)
Symbol to graph	0.12 (26)	0.24 (25)
Symbol to table	0.00 (22)	0.15 (27)
Symbol to word	0.00 (27)	0.16 (25)
Word to graph	0.00 (27)	0.08 (25)
Word to symbol	0.18 (22)	0.07 (27)
Word to table	0.00 (26)	0.24 (25)

As seen in Table 3, overall performance was quite low on the translation pretest, and remained relatively low at posttest. On linear problems, at pretest, student performance differed from zero (with alpha = .10) only on graph-to-table, word-expression-to-graph, word-expression-to-symbol, and word-expression-to-table translations. At posttest, performance differed significantly from zero on all of these same input-out pairs as well as on symbol-to-table and symbol-to-word-expression translations. On nonlinear problems, performance began at and remained near zero regardless of input-output pair, with performance differing significantly from zero only on symbol-to-graph translations at pretest and on graph-to-table, symbol-to-graph, symbol-to-word-expression, and word-expression-to-table translations at posttest.

Figure 3 presents posttest performance separately for each input-output pair for students in each group (collapsed across levels of linearity).

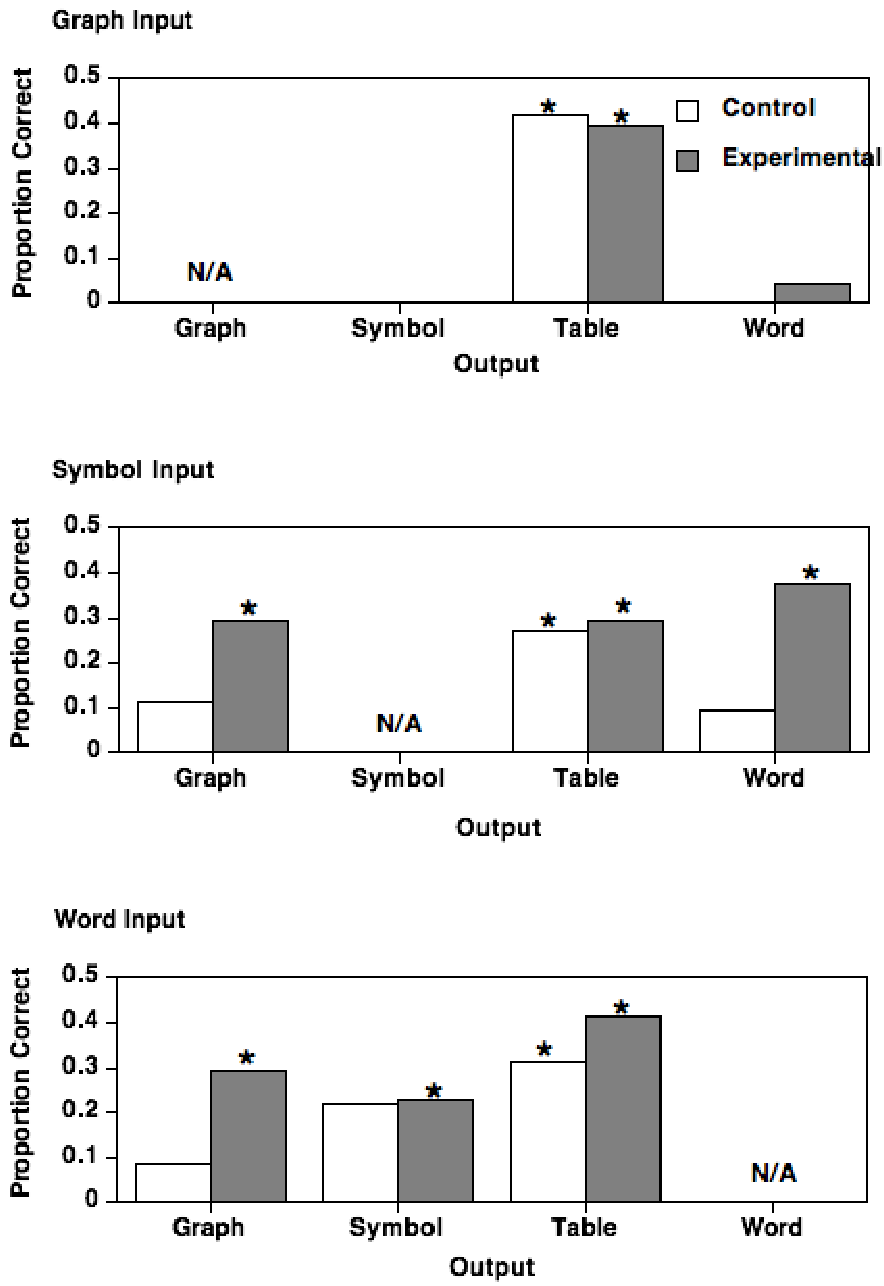

Figure 3. **Students' posttest success producing graph, symbol, table, and word expression outputs given (a) graphs, (b) symbols, and (c) word expressions as inputs. Starred (*) columns indicate scores that are significantly different from zero.**

As seen in Figure 3, at posttest students in the CM group demonstrated skills for translating from any of the input representations to a table of values. Students in the BI group demonstrated even broader skills—specifically, skills for translating from any of the input representations to tables of values and to graphs, as well as translating between word expressions and symbols. This finding suggests an emerging reciprocity between words and symbols that may be based on their mutual compatibility as holistic and propositional representations.

Discussion

In this section, we interpret the empirical findings in light of our research questions and discuss the broader implications of representation use for students' reasoning and conceptual development in mathematics.

The Bridging Instruction Curriculum

Instruction in both CM and BI led to measurable gains in students' algebra problem solving. Gains were small for graph-based problems, most likely because students had little room for improvement on average, given their strong pretest performance (Figure 1). Gains with symbolically presented problems were the largest and showed that both instructional approaches improved students' abilities to solve problems using equations.

Along with these similarities, some important differences were evident in the outcome measures of students who used the different curricula. BI students showed larger gains overall than CM students, even though all the students were taught by the same teacher. Gains made by CM students were confined to linear, symbolic problem solving (from 28.9% to 46.8%, a 17.9-point gain). BI students showed comparable gains in linear, symbolic problem solving (from 19.0% to 39.2%, a 20-point gain) but also showed superior gains in linear problem solving overall (collapsing across representational formats, +10.1%), as well as significant gains in nonlinear problem solving (collapsing across representational formats, +11.8%). The data also show that, whereas students in the CM group showed improvement exclusively in use of symbolic representations, students in the BI condition showed significantly greater problem-solving gains using word expressions, particularly for nonlinear problems (nonlinear CM word gain = -1.5%, nonlinear BI word gain = 22.8%), as well as gains in symbol use.

Turning to possible hypotheses that account for these findings, we first propose that teaching students about linear and nonlinear functions in an integrated fashion is beneficial to their learning of each. A variation of this hypothesis is that spending more time on nonlinear functions may advance the learning of the less complex linear functions because many elements of conceptual and procedural knowledge of linearity may be enlisted and further developed during tasks involving nonlinear functions (cf. Bahrick & Hall, 1991). It is also plausible that by explicitly connecting algebraic representations and procedures to students' invented strategies and representations, BI helps students construct a conceptual grounding for the meaning of these representations that supports greater fluency. This would corroborate and extend prior studies of early algebra learning that built explicitly on students' invented strategies and representations for algebra story problem solving (e.g., Nathan & Koedinger, 2000). Because of the design-based nature of the treatment comparisons, many factors differed between the two instructional conditions. Consequently, the present study does not allow us to make any strong claims about the individual versus combined effects of connecting to students' prior mathematical ideas or of integrating the learning of linear and nonlinear functions. Further studies of the contributions of each of these toward the development of students' representational fluency are needed to be more conclusive.

Translation Across Mathematical Representations

The data on translation tasks make it clear that this aspect of representational fluency is an advanced skill: Students struggled to move among tabular, graphical, verbal, and symbolic forms of representation far more than they struggled to use individual representational formats to solve problems. This difficulty is likely exacerbated by the need for students to generate the target representation from a given representation in order to perform the translation. At pretest, performance was essentially at zero for all but a few types of translation problems. Students could translate from verbal rules to symbolic equations (performing at about 30%) and from graphs to tables (at about 20%). Students were more adept at translating linear relations than nonlinear relations, but there was tremendous room for improvement in all the areas of representational translation that we explored.

These data point to an important difference in the psychological demands of comprehending a representation and producing one. Of particular interest here is students' differential performance in graph use between translation and problem solving. Students could use preconstructed graphs at about the 80% level for problem solving. However, student performance levels dropped to the 12.6% level when they had to produce graphs depicting the same quantitative relations. Tasks with preconstructed representations that serve as effective memory cues allow students to access knowledge that would otherwise go unretrieved (cf. Tulving & Thompson, 1973). This is a potentially important consideration, since the differential demands of problem-solving tasks and production-intensive translation tasks might be overlooked by practitioners, curriculum developers, and assessment designers.

At posttest, students in both curricular groups succeeded on problems that required them to produce tables of values from words, graphs, or equations. In addition, BI students successfully produced graphs from word expressions and symbolic equations. Gains on symbol and word output tasks were relatively minimal. The patterns of success suggest that attributes of the representations themselves are important factors in students' learning.

One interpretation of these data is that students gained most when they could generate representations that were instance-based (i.e., pointwise) or local in their representational scope. Tables of values naturally have this instance-based character, as reflected by the scoring criteria. Consequently, successful performance can be achieved by specifying independent points through predominantly arithmetic means. In contrast, students struggled with more holistic representations, such as verbal rules and symbolic equations. Holistic representations like equations require students to abstract from specific instances and capture the covariation of a function for all values (in this case, the infinitely large set of all real numbers) in a concise, unified, and syntactically accurate description (Nathan & Kim, 2007).

Graphs present an interesting case as they can be local in character (as with scatter plots and bar graphs) or holistic (as with line graphs). To further understand performance with representations along this dimension, we performed an additional analysis of BI students' post-intervention ability to produce proper graphs. The students exhibited relatively high levels of performance when their graphs were evaluated using instance-based scoring criteria (M = 29.5%) but much lower performance when the graphs were evaluated using holistic criteria (M = 4.3%).

These findings are consistent with the theory of developmental progression in children's understanding of mathematical functions proposed by Kalchman and colleagues (Kalchman, 1998; Kalchman et al., 2001). Kalchman (1998) held that procedurally based (i.e., computational) representations, such as tables of instances, and analogical representations of mathematical functions, such as bar graphs, developmentally precede and form the basis for the more integrative, geometric representations, such as line graphs. From this, one would expect to see greater success with instance-based graph and table output problems, as shown in the current investigation. Our work also shows that graphs can be used in either an instance-based or holistic fashion, which follows from their twofold nature as integrative representations. This duality is not as apparent with symbolic, word, or table representations.

A plausible hypothesis is that tables and instance-based graphs may be natural entry points into the mathematics of covariation, which serves as a central idea for the mathematics of functions. Graphs may be particularly effective in helping students make the transition to more holistic representations since they support both modalities. Ultimately, graphs may help students learn symbolic formalisms and reap the rewards they can provide in the context of increasingly complex relationships, as we explore in the next section.

The Representation-Complexity Trade-Off in Problem Solving

Students' problem-solving performance, both before and after the intervention, was heavily influenced by the specific algebraic representation, even when the underlying quantitative structure was kept constant. In particular, problem-solving performance using graphical representations (85.7%) exceeded that of all of the other representations (39% for words and 22.4% for symbols). This graphical advantage held for both linear and nonlinear functions regardless of the slope sign of the function.

With regard to the relationship between representation and linearity, pretest data on linear problems showed that students achieved greater success when the problems were presented in a verbal format than when they were presented symbolically (linear word pretest M = 65.1%, linear symbol pretest M = 24.0%). However, when problems dealt with nonlinear relations, symbols proved more effective than verbal representations (nonlinear word pretest M = 13.0%, nonlinear symbol pretest M = 20.8%). As noted, this result replicates earlier findings showing a complexity-based trade-off among representations (Koedinger et al., 2008). This interaction between complexity and representation appears to be due in part to the fact that verbally presented problems tend to elicit inefficient but highly reliable arithmetic-based solution strategies (such as "guess and test" and working backwards) that work well with low-complexity relations (Koedinger & Nathan, 2004). With increasing complexity, however, the arithmetic strategies become computationally unwieldy, whereas symbol-based representations adequately "scale up" to the greater representational demands. This empirical finding suggests the historical rationale for the movement away from purely linguist representations of mathematical procedures and structures and the development and widespread adoption of mathematical formalisms, as the nature of the phenomena under study became more complex.

The current study replicates but also extends the representation-complexity interaction by showing that graphs “scale” nicely with the shift in complexity (Figure 1). This finding makes sense as reading a point off a graph is not much more demanding for the nonlinear graphs used in the assessment than it is for the linear graphs. The finding also underscores the utility of graphical representations across a broad range of mathematical relations.

Conclusions

The present study investigated beginning algebra students’ abilities to work within and translate among various representational formats. On the whole, students were more successful at solving problems with a single representation than at translating among representations. The data on students’ performance suggest that significant gaps exist between students’ abilities to comprehend particular representations and their abilities to produce those representations. However, performance strongly depended on the particular representational format. Overall performance improved with instruction, with the largest and broadest gains achieved by students in the *Bridging Instruction* curriculum, which focused from the outset on both linear and nonlinear relations and made explicit links from students’ invented strategies and representations to more standardized representations and solution methods. Finally, students appeared to attain fluency with instance-based representations (such as tables and pointwise graphs) before they attained fluency with more global, holistic representations (such as symbolic equations and verbal expressions). These findings highlight the challenges of fostering representational fluency in early algebra instruction and indicate areas for research on students’ mathematical reasoning and future curricular development.

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Appendix

Table A1
Example of Multiple Forms of a Linear Problem Used in the Assessment

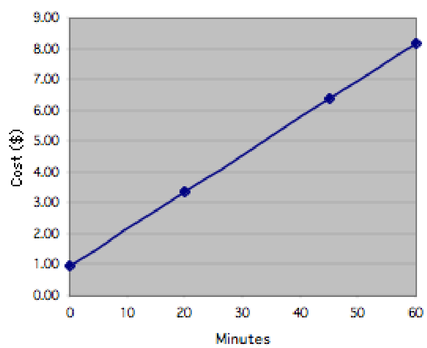
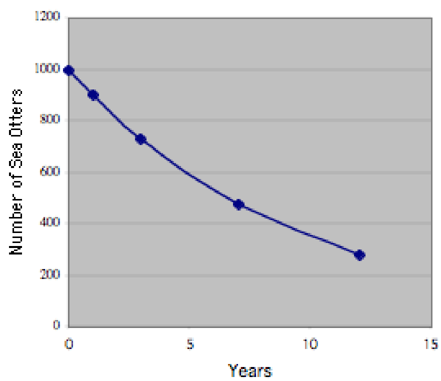
Problem section	**Information presented**
Situation introduced	Cassandra sells phone cards to college students so they can make long distance calls for a good price. Each card has a base charge and a per-minute rate.
Input presented	
Graph input	Below is a graph you can use to find the price of the card if you know the number of minutes on it. 
Symbol input	The expression below shows how to find the price of the card, *p*, if you know the number of minutes on it, *n*. $p = 0.99 + 0.12n$
Word expression input	The description below tells you how to find the price of the card if you know the number of minutes on it. To find the price of the card, you multiply the number of minutes by the per-minute rate of $0.12 and then add the base charge of $0.99.
Part a (problem solving)	What would be the price of a card with 30 minutes?
Part b (problem solving)	How many minutes would be on a card that cost $6.99?
Part c (translation)	
Graph output	Make a graph that you can use to find the price of the card if you know the number of minutes. (*Examination packets included graph paper.*)
Symbol output	Write a mathematical expression that tells how to find the price of the card if you know the number of minutes.
Table output	Make a table of values that you can use to find the price of the card if you know the number of minutes.
Word expression output	Describe in words how to find the price of the card if you know the number of minutes.

Table A2

Example of Multiple Forms of Nonlinear Problems Used in the Assessment

Problem section	Information presented
Situation introduced	Marine biologists are concerned that the population of sea otters is rapidly decreasing in one area.
Input presented	
Graph input	Below is a graph that you can use to find the population of sea otters in that area, if you know the number of years since the study began. 
Symbol input	The expression below shows how to find the population of sea otters in that area, *p*, if you know the number of years since the study began, *n*. $p = 1{,}000(1 - .10)^n$
Word expression input	To find the population, you take 1 minus 0.10 and raise it to the power of the number of years since the study began, and then you take the result and multiply it by the starting population of 1,000.
Part a (problem solving)	What was the population of sea otters after 5 years of the study?
Part b (problem solving)	What was the population of sea otters when the study began?
Part c (translation)	
Graph output	Make a graph that you can use to find the population of sea otters if you know the number of years since the study began. (*Examination packets included graph paper.*)
Symbol output	Write a mathematical expression that tells how to find the population of sea otters if you know the number of years since the study began.
Table output	Make a table of values that you can use to find the population of sea otters if you know the number of years since the study began.
Word expression output	Describe in words how to find the population of sea otters if you know the number of years since the study began.